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Discrete fractional Sobolev norms for domain decomposition preconditioning

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We present a new approach for preconditioning the interface Schur complement arising in the domain decomposition of second-order scalar elliptic problems. The preconditioners are discrete interpolation norms recently introduced in Arioli & Loghin (2009, Discrete interpolation norms with applications. *SIAM J. Numer. Anal.*, **47**, 2924–2951). In particular, we employ discrete representations of norms for the Sobolev space of index 1/2 to approximate the Steklov–Poincaré operators arising from nonoverlapping one-level domain decomposition methods. We use the coercivity and continuity of the Schur complement with respect to the preconditioning norm to derive mesh-independent bounds on the convergence of iterative solvers. We also address the case of nonconstant coefficients by considering the interpolation of weighted spaces and the corresponding discrete norms.

Keywords: fractional Sobolev norms; interface preconditioners; domain decomposition; generalized Lanczos algorithm; square-root Laplacian.

1. Introduction

The key ingredient in any domain decomposition method is the ability to solve an interface problem involving a pseudodifferential operator, generally referred to as the Steklov–Poincaré operator. Currently, there are two approaches to solving this problem. The most popular is an implicit approach that approximates the interface problem, which is a lower-dimensional problem, by solving a suitable sequence of subdomain problems. The original Schwarz method (Schwarz, 1869) used this implicit approach and essentially most domain decomposition algorithms fall in the same category: Dirichlet–Neumann, Neumann–Neumann, additive and multiplicative Schwarz methods, FETI methods, balancing domain decomposition methods, etc. For this class of problems, independence of discretization and decomposition parameters does not generally hold and further consideration of the problem is needed. Some notable approaches are solving the subdomains with modified boundary conditions, the introduction of overlap and coarse-grid corrections. For reviews, descriptions and analyses see Chan & Mathew (1994), Quarteroni & Valli (1999) and Toselli & Widlund (2005).

The second approach aims to address the interface problem directly. Several issues need to be addressed in this case. Firstly, the interface operator is only known for some simple problems with constant coefficients. Secondly, any suitable discretizations will give rise to dense matrices that are not amenable to sparse matrix preconditioning. Finally, just as for the implicit approach, we may require a coarse-grid correction technique to achieve independence of the decomposition parameter. However, the second approach appears to generally lead to solution techniques independent of the mesh parameter. Some early contributions that consider simple model problems on uniform meshes and regular decompositions can be found in Dryja (1982), Golub & Mayers (1984), Bramble *et al.* (1986), Widlund (1988), Chan (1987), Chan & Mathew (1992) and Ovtchinnikov (1993). Mesh independence remains the most important criterion for choosing a domain decomposition technique, given the current suboptimal scalability of parallel computers. The solution method we consider below falls into this category.

In this work we present an alternative for preconditioning the Schur complement arising from DD problems, which has not been considered to date. In particular, we exploit directly the well-known fact that the Steklov–Poincaré operators induce bilinear forms that are continuous and coercive on Sobolev spaces of index 1/2. More precisely, we construct finite element representations of norms for these interpolation spaces which we then use to precondition the problem. These norms can be interpreted as discrete representations of square-root Laplacians defined on the interface. Our approach is related to that of Bramble *et al.* (1986) (see also Dryja, 1982; Golub & Mayers, 1984; Bjørstad & Widlund, 1986 and Chan, 1987), who considered square-root Laplacians constructed on each face of the interface for preconditioning the problem. We also mention here the work of Bernardi *et al.* (2008) who introduce an interface preconditioner with performance that is independent of the size of the problem via a mixed finite element formulation. Many of the cited approaches produce local nonsparse operators that necessarily lead to deterioration in performance. Furthermore, often, only the case of uniform discretizations on straight edges is considered, leaving as an open problem the generalization to arbitrary subdivisions and decompositions of the computational domain.

The paper is structured as follows. Section 2 contains the problem description together with some standard functional analytic results, while Section 3 describes the discrete setting. Section 4 introduces the preconditioners for the Steklov–Poincaré operator together with the convergence analysis for preconditioned GMRES. In the same section we include a description of a sparse implementation of our preconditioners. Finally, Section 5 contains a range of numerical experiments both in two and three dimensions, which validate our analysis.

2. Problem description

We review below the standard formulation of nonoverlapping domain decomposition problems for a general scalar elliptic problem.

2.1 *Notation and definitions*

Throughout the paper we will use the following notation and standard results. Given an open, simply connected domain U in $\mathbb{R}^d$, its boundary will be denoted by ∂U. We denote by $\mathcal{C}_0^\infty(U)$ the space of infinitely differentiable functions defined on U with compact support in U. We will also denote by $L^2(U)$ the Lebesgue space of square-integrable functions defined on U endowed with inner product $(\cdot, \cdot)$, and by $H^m(U)$ the Sobolev space of order m equipped with norm $\|\cdot\|_{m,U}$ and seminorm $|\cdot|_{m,U}$ with the convention $H^0(U) = L^2(U)$.

Let $Q \subset \partial U$. The Sobolev spaces of real index $0 \leqslant s \leqslant m$ are defined as interpolation spaces of index $\theta = 1 - s/m$ for the pair $[H^m(Q), L^2(Q)]$,

$$H^s(Q) := [H^m(Q), L^2(Q)]_\theta, \quad \theta = 1 - s/m.$$

For any s, the space $H_0^s(Q)$ denotes the completion of $C_0^\infty(Q)$ in $H^s(Q)$ (see, e.g., Lions & Magenes, 1968, p. 60). In particular, we shall be interested in the interpolation space

$$H^{1/2}(Q) = [H^1(Q), L^2(Q)]_{1/2}$$

for which there holds $H_0^{1/2}(Q) \equiv H^{1/2}(Q)$. Another space of interest is $H_{00}^{1/2}(Q)$ which is a subspace of $H_0^{1/2}(Q)$ and is defined as the interpolation space of index 1/2 for the pair $[H_0^1(Q), L^2(Q)]$,

$$H_{00}^{1/2}(Q) = [H_0^1(Q), H^0(Q)]_{1/2}.$$

Norms on $H^{1/2}(Q)$, $H_{00}^{1/2}(Q)$ will be denoted by the same notation: $|\cdot|_{1/2,Q}$ or $\|\cdot\|_{1/2,Q}$, with the assumption that it is evident from the context which space is under consideration. We will return to the definition of these norms in Section 3. The dual of $H_{00}^{1/2}(Q)$ is denoted by $(H_{00}^{1/2}(Q))' \subset H^{-1/2}(Q)$, where $H^{-1/2}(Q) := (H^{1/2}(Q))' \equiv (H_0^{1/2}(Q))'$. The duality between $H_{00}^{1/2}(Q)$ and its dual will be denoted by $\langle\cdot,\cdot\rangle$.

Finally, we will make use of the trace operator γ_0: $H^1(U) \to H^{1/2}(\partial U)$ which is known to be surjective and continuous, i.e., there exists a constant $c_\gamma(U)$ such that

$$\|\gamma_0 v\|_{1/2,\partial U} \leqslant c_\gamma(U)\|v\|_{1,U} \quad \forall v \in H^1(U). \tag{2.1}$$

Let

$$H^1_{\partial U\setminus Q}(U) := \{v \in H^1(U) \mid \gamma_0 v = 0 \text{ on } \partial U \setminus Q\}.$$

A similar inequality holds if we take γ_0: $H^1_{\partial U\setminus Q}(U) \to H_{00}^{1/2}(Q)$:

$$\|\gamma_0 v\|_{1/2,Q} \leqslant c_\gamma(U)\|v\|_{1,U} \quad \forall v \in H^1_{\partial U\setminus Q}(U). \tag{2.2}$$

We will also assume that the following Poincaré inequality holds:

$$\|v\|_{0,U} \leqslant C_P(U)|v|_{1,U}. \tag{2.3}$$

2.2 *Domain decomposition for scalar elliptic partial differential equations*

Let now Ω denote an open subset of $\mathbb{R}^d$ with boundary $\partial\Omega$ and consider the problem

$$\begin{cases} \mathcal{L}u = -\mathrm{div}(\mathbf{a}\nabla u) + \vec{b}\cdot\nabla u + cu = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.4}$$

where $f \in L^2(\Omega)$, $c \in L^\infty(\Omega)$, $\vec{b}$ is a vector function whose entries are Lipschitz continuous real-valued functions on $\bar{\Omega}$ and a is a symmetric $d \times d$ matrix whose entries are bounded, piecewise continuous real-valued functions defined on $\bar{\Omega}$, with

$$0 < a_{\min} \leqslant \zeta^{\mathrm{T}} a(\mathbf{x})\zeta \leqslant a_{\max} \quad \forall \zeta \in \mathbb{R}^d, \text{ a.e. } \mathbf{x} \in \bar{\Omega}. \tag{2.5}$$

We will also assume that the following standard condition holds:

$$c - \frac{1}{2}\nabla \cdot \vec{b} \geqslant c_{\min} > 0 \quad \text{a.e. } \mathbf{x} \in \Omega. \tag{2.6}$$

The weak formulation of problem (2.4) reads

$$\begin{cases} \text{find } u \in H_0^1(\Omega) \text{ such that for all } v \in H_0^1(\Omega), \\ B(u, v) = (f, v), \end{cases} \tag{2.7}$$

where the bilinear form $B(\cdot, \cdot) : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ is defined via

$$B(v, w) = (a \cdot \nabla v, \nabla w) + (\vec{b} \cdot \nabla v + cv, w).$$

Let

$$\bar{\Omega} = \bigcup_{i=1}^{N} \bar{\Omega}_i, \quad \Omega_i \cap \Omega_j \equiv \emptyset \ (i \neq j),$$

and let $\Gamma \subset \mathbb{R}^{d-1}$ denote the set of internal boundaries associated with the above partition of Ω,

$$\Gamma = \bigcup_{i=1}^{N} \Gamma_i \quad (\Gamma_i := \partial\Omega_i \setminus \partial\Omega).$$

Given a function v defined on Ω we will denote by v_i the restriction of v to Ω_i: $v_i = v\,|_{\Omega_i}$. With this notation, we define the bilinear forms $B_i(\cdot, \cdot) : H^1_{\partial\Omega\cap\partial\Omega_i}(\Omega_i) \times H^1_{\partial\Omega\cap\partial\Omega_i}(\Omega_i) \to \mathbb{R}$, similar to $B(\cdot, \cdot)$:

$$B_i(v_i, w_i) = (a_i \cdot \nabla v_i, \nabla w_i) + (\vec{b}_i \cdot \nabla v_i + c_i v_i, w_i).$$

Let now $v \in H_0^1(\Omega)$. Then $v_i = v\,|_{\Omega_i} \in H^1_{\partial\Omega\cap\partial\Omega_i}(\Omega_i)$ and there holds

$$B(u, v) = \sum_{i=1}^{N} B_i(u_i, v_i), \quad (f, v) = \sum_{i=1}^{N} (f, v_i). \tag{2.8}$$

Let u denote the solution of (2.4) and let $u_i = u\,|_{\Omega_i}$, $u_i\,|_{\Gamma_i} = \lambda_i$. Let now $\eta \in H_{00}^{1/2}(\Gamma)$ with $\eta_i = \eta\,|_{\Gamma_i}$ and let v_i be the solution to the problem

$$\begin{cases} \mathcal{L}v_i = 0 & \text{in } \Omega_i, \\ v_i = 0 & \text{on } \partial\Omega_i \setminus \Gamma_i, \\ v_i = \eta_i & \text{on } \Gamma_i. \end{cases} \tag{2.9}$$

The function v defined via $v\,|_{\Omega_i} = v_i$ is a generalized $\mathcal{L}$-extension of the function $\eta \in H_{00}^{1/2}(\Gamma)$ to the domain Ω and will be denoted by $E\eta$ and $E_i\eta_i$, respectively. Any other extensions will be denoted by $F\eta$ and $F_i\eta_i$.

We will also need the following elliptic regularity result which is known to hold for the weak solution of (2.9) (see, e.g., Agmon *et al.*, 1959)

$$\|v_i\|_{1,\Omega_i} = \|E_i\eta_i\|_{1,\Omega_i} \leqslant C_e\|\eta_i\|_{1/2,\Gamma_i}. \tag{2.10}$$

We define the *Steklov–Poincaré operator* $\mathcal{S}$: $H_{00}^{1/2}(\Gamma) \to (H_{00}^{1/2}(\Gamma))'$ as follows. Let $\eta, \mu \in H_{00}^{1/2}(\Gamma)$ with $\eta\,|_{\Gamma_i} =: \eta_i$, $\mu\,|_{\Gamma_i} =: \mu_i$. We define $\mathcal{S}$ via

$$\langle \mathcal{S}\eta, \mu\rangle = \sum_{i=1}^{N} \int_{\Gamma_i} \mathbf{n}_i \cdot a \cdot \nabla(E\eta_i)\mu_i \, \mathrm{d}s(\Gamma_i) =: \sum_{i=1}^{N} \langle \mathcal{S}_i\eta_i, \mu_i\rangle. \tag{2.11}$$

Using integration by parts, the operator $\mathcal{S}$ can be given the following alternative representation:

$$\langle \mathcal{S}\eta, \mu\rangle = B(E\eta, F\mu) = \sum_{i=1}^{N} B_i(E_i\eta_i, F_i\mu_i) \quad \forall\, \eta, \mu \in H_{00}^{1/2}(\Gamma). \tag{2.12}$$

With this definition of $\mathcal{S}$ our problem can be recast as an ordered sequence of three decoupled sets of problems involving the same operator $\mathcal{L}$ with essential boundary conditions on each subdomain together with a problem set on the interface Γ.

$$\begin{aligned}
&\text{(i)} \quad \begin{cases} \mathcal{L}u_i^{\{1\}} &= f \quad \text{in } \Omega_i, \\ u_i^{\{1\}} &= 0 \quad \text{on } \partial\Omega_i, \end{cases} \\
&\text{(ii)} \quad \begin{cases} \mathcal{S}\lambda &= -\displaystyle\sum_{i=1}^{N} \mathbf{n}_i \cdot a \cdot \nabla u_i^{\{1\}} \quad \text{on } \Gamma, \end{cases} \\
&\text{(iii)} \quad \begin{cases} \mathcal{L}u_i^{\{2\}} &= 0 \quad \text{in } \Omega_i, \\ u_i^{\{2\}} &= \lambda_i \quad \text{on } \Gamma_i. \\ u_i^{\{2\}} &= 0 \quad \text{on } \partial\Omega_i \setminus \Gamma_i. \end{cases}
\end{aligned} \tag{2.13}$$

The resulting solution is

$$u|_{\Omega_i} = u_i^{\{1\}} + u_i^{\{2\}}.$$

We now turn to the properties of the interface operator $\mathcal{S}$. Given representation (2.11) we can immediately see that $\mathcal{S}$ is nonsymmetric unless $\vec{b} = \mathbf{0}$. One can show further that $\mathcal{S}$ is a bounded positive operator on $H_{00}^{1/2}(\Gamma)$.

LEMMA 2.1 Let $\mathcal{S}$ be defined by (2.11) and let (2.6) hold. Then there exist constants α_1, α_2 such that for all $\eta, \mu \in H_{00}^{1/2}(\Gamma)$

$$\alpha_1 \|\eta\|_{1/2,\Gamma}^2 \leqslant \langle S\eta, \eta\rangle, \quad \langle S\eta, \mu\rangle \leqslant \alpha_2 \|\eta\|_{1/2,\Gamma} \|\mu\|_{1/2,\Gamma}.$$

Proof. Let $v_i = E_i\eta_i$, $w_i = E_i\mu_i$ satisfy (2.9), so that $v_i, w_i \in H^1_{\partial\Omega_i\setminus\Gamma_i}(\Omega_i)$. We have, using (2.6),

$$\begin{aligned}
\langle \mathcal{S}_i\eta_i, \eta_i\rangle &= B_i(v_i, v_i) \\
&= (a \cdot \nabla v_i, v_i) + (\vec{b} \cdot \nabla v_i, v_i) + (cv_i, v_i) \\
&= (a \cdot \nabla v_i, v_i) + \left(\left(c - \frac{1}{2}\nabla \cdot \vec{b}\right) v_i, v_i\right) \\
&\geqslant a_{\min} |v_i|_{1,\Omega_i}^2 + c_{\min} \|v_i\|_{0,\Omega_i}^2 \\
&\geqslant \min\{a_{\min}, c_{\min}\} \|v_i\|_{1,\Omega_i}^2.
\end{aligned}$$

Moreover, using the Poincaré inequality (2.3) we get

$$\begin{aligned}\langle \mathcal{S}_i \eta_i, \mu_i \rangle &= B_i(v_i, w_i)\\ &\leqslant a_{\max}|v_i|_{1,\Omega_i}|w_i|_{1,\Omega_i} + \|\vec{b}\|_{L^\infty(\Omega_i)}|v_i|_{1,\Omega_i}\|w_i\|_{0,\Omega_i} + \|c\|_{L^\infty(\Omega_i)}\|v_i\|_{0,\Omega_i}\|w_i\|_{0,\Omega_i}\\ &\leqslant \max\{a_{\max} + \|\vec{b}\|_{L^\infty(\Omega_i)}C_P(\Omega_i), \|c\|_{L^\infty(\Omega_i)}\}\|v_i\|_{1,\Omega_i}\|w_i\|_{1,\Omega_i}.\end{aligned}$$

Since $\gamma_0 v_i = \eta_i$, $\gamma_0 w_i = \mu_i$, the trace inequalities (2.1) and (2.2) read, for all $i = 1, \ldots, N$,

$$\|\eta_i\|_{1/2,\Gamma_i} \leqslant C_\gamma(\Omega_i)\|v_i\|_{1,\Omega_i}, \quad \|\mu_i\|_{1/2,\Gamma_i} \leqslant C_\gamma(\Omega_i)\|w_i\|_{1,\Omega_i}.$$

The result follows from regularity estimate (2.10) and definition (2.11) of the operator $\mathcal{S}$. □

2.3 *Finite element discretisations*

In order to write down the weak formulation of problems (2.13) we rewrite the set of equations (2.13, (iii)) as

$$\text{(iii)} \begin{cases} \mathcal{L}\tilde{u}_i^{\{2\}} = -\mathcal{L}z_i & \text{in } \Omega_i,\\ \tilde{u}_i^{\{2\}} = 0 & \text{on } \Gamma_i.\\ \tilde{u}_i^{\{2\}} = 0 & \text{on } \partial\Omega_i \setminus \Gamma_i. \end{cases}$$

where $\tilde{u}_i^{\{2\}} = u_i^{\{2\}} - z$. Let $P_r(\mathfrak{t})$ denote the space of polynomials in d variables of degree r defined on a set $\mathfrak{t} \subset \mathbb{R}^d$. Let

$$V_i^h = V_i^{h,r} := \{w \in C^0(\Omega_i)\colon w|_{\mathfrak{t}} \in P_r\ \forall \mathfrak{t} \in \mathfrak{T}_h,\ w\ |_{\partial\Omega\cap\partial\Omega_i} = 0\} \subset H^1_{\partial\Omega\cap\partial\Omega_i}(\Omega_i) \tag{2.14}$$

be a finite-dimensional space of piecewise polynomial functions defined on some subdivision $\mathfrak{T}_h$ of Ω into simplices $\mathfrak{t}$ of maximum diameter h. Let further $V_{iI}^h, V_{iB}^h \subset V_i^h$ satisfy $V_{iI}^h \oplus V_{iB}^h \equiv V_i^h$. Let also

$$V_{iI}^h = \operatorname{span}\{\phi_k^i, k = 1, \ldots, n_i^I\}, \quad V_{iB}^h = \operatorname{span}\{\psi_k^i, k = 1, \ldots, n_i^B\},$$

and set $n_I = \sum_i n_i^I$ and $n_B = \sum_i n_i^B$. Let further

$$V_B^h = \bigcup_{i=1}^N V_{iB}^h$$

and let $\{\psi_k\}_{1\leqslant k\leqslant n_B}$ denote a basis for V_B^h. Let $S^h = \operatorname{span}\{\gamma_0(\Gamma)\psi_k\}_{1\leqslant k\leqslant n_B}$ and set

$$V^h = \bigcup_{i=1}^N V_i^h \subset H_0^1(\Omega).$$

The finite element discretisation of the weak formulation (2.7) reads

$$\begin{cases} \text{find } u_h \in V^h \text{ such that for all } v_h \in V^h,\\ B(u_h, v_h) = (f, v_h). \end{cases} \tag{2.15}$$

The finite element discretisations of formulation (2.13) are as follows.

$$\begin{aligned}
&\text{(i)} \quad \begin{cases} \text{Find } u_{hi}^{\{1\}} \in V_{iI}^h \text{ such that for all } v_{hi} \in V_{iI}^h, \\ B_i\big(u_{hi}^{\{1\}}, v_{hi}\big) = (f_i, v_{hi}). \end{cases} \\
&\text{(ii)} \quad \begin{cases} \text{Find } \lambda_h \in S^h \text{ such that for all } \eta_h \in S^h, \\ s(\lambda_h, \eta_h) = \sum_{i=1}^{N} \big[(f_i, F_i\eta_{hi}) - B_i\big(u_{hi}^{\{1\}}, F_i\eta_{hi}\big)\big]. \end{cases} \\
&\text{(iii)} \quad \begin{cases} \text{Find } \tilde{u}_{hi}^{\{2\}} = u_{hi}^{\{2\}} - z_{hi} \in V_{iI}^h \text{ such that for all } v_{hi} \in V_{iI}^h, \\ B_i\big(\tilde{u}_{hi}^{\{2\}}, v_{hi}\big) = -B_i(z_{hi}, v_{hi}). \end{cases}
\end{aligned} \tag{2.16}$$

2.4 *Matrix formulation*

Let

$$A\mathbf{u} = \begin{pmatrix} A_{II} & A_{IB} \\ A_{BI} & A_{BB} \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ \mathbf{u}_B \end{pmatrix} = \begin{pmatrix} \mathbf{f}_I \\ \mathbf{f}_B \end{pmatrix} = \mathbf{f} \tag{2.17}$$

represent the linear system associated with the discrete formulation (2.15) with $A \in \mathbb{R}^{n\times n}, \mathbf{u} \in \mathbb{R}^n$, where $n = n_I + n_B$ and $A_{II} \in \mathbb{R}^{n_I\times n_I}$, $A_{IB}, A_{BI}^{\mathrm{T}} \in \mathbb{R}^{n_I\times n_B}$, $A_{BB} \in \mathbb{R}^{n_B\times n_B}$ are given by

$$A_{II} = \begin{pmatrix} A_{II}^1 & & & & \\ & \ddots & & & \\ & & A_{II}^i & & \\ & & & \ddots & \\ & & & & A_{II}^N \end{pmatrix}, \quad A_{IB} = \begin{pmatrix} A_{IB}^1 \\ \vdots \\ A_{IB}^i \\ \ddots \\ A_{IB}^N \end{pmatrix},$$

$$A_{BI} = \begin{pmatrix} A_{BI}^1 & \cdots & A_{BI}^i & \cdots & A_{BI}^N \end{pmatrix},$$

with

$$\begin{aligned} (A_{II}^i)_{kk} &= B_i(\phi_k^i, \phi_k^i), \quad (A_{IB}^i)_{kj} = B_i(\phi_k^i, \psi_j^i), \\ (A_{BI}^i)_{jk} &= B_i(\psi_l^i, \phi_l^i), \quad (A_{BB})_{ll} = B(\psi_l, \psi_l), \end{aligned}$$

for all $k = 1, \ldots, n_i^I$, $j = 1, \ldots, n_i^B$, $l = 1, \ldots, n_B$.
With this notation, one can show that formulation (2.16) has the following matrix representation:

$$\text{(i) } A_{II}\mathbf{u}_I^{\{1\}} = \mathbf{f}_I, \quad \text{(ii) } S\mathbf{u}_B = \mathbf{f}_B - A_{BI}\mathbf{u}_I^{\{1\}}, \quad \text{(iii) } \mathbf{u}_I^{\{2\}} = -A_{II}^{-1}A_{IB}\mathbf{u}_B,$$

where S is the matrix representation of $s(\cdot,\cdot)$ in the basis of S^h. Furthermore, S can be shown to be the Schur complement of A_{BB} in A:

$$S = A_{BB} - A_{BI}A_{II}^{-1}A_{IB}.$$

It follows that equations (i–iii) represent a Schur complement approach for the original linear system (2.17) with global solution $\mathbf{u}$ given by

$$\mathbf{u} = \begin{pmatrix} \mathbf{u}_I \\ \mathbf{u}_B \end{pmatrix} = \begin{pmatrix} \mathbf{u}_I^{\{1\}} \\ \mathbf{u}_B^{\{1\}} \end{pmatrix} + \begin{pmatrix} \mathbf{u}_I^{\{1\}} \\ \mathbf{u}_B^{\{1\}} \end{pmatrix}.$$

3. Preconditioners for the Steklov–Poincaré operator

The result of Lemma 2.1 holds also in the discrete case for the choice of space S_h introduced in Section 2. In particular, it translates into the following coercivity and continuity bounds for $s(\cdot,\cdot) : S_h \times S_h$.

LEMMA 3.1 Let $s(\cdot,\cdot)$ be defined as in (2.16) and let (2.6) hold. Then there exist constants α_1, α_2 such that for all $\eta_h, \mu_h \in S_h \subset H_{00}^{1/2}(\Gamma)$,

$$\alpha_1 \|\eta_h\|_{1/2,\Gamma}^2 \leqslant s(\eta_h, \eta_h), \quad s(\eta_h, \mu_h) \leqslant \alpha_2 \|\eta_h\|_{1/2,\Gamma} \|\mu_h\|_{1/2,\Gamma}.$$

In order to derive the corresponding matrix formulation of the above result we need to recall the results in Arioli & Loghin (2009).

3.1 *Abstract interpolation results*

Let X, Y denote two Hilbert spaces with $X \subset Y$, X dense and continuously embedded in Y. Let $\langle\cdot,\cdot\rangle_X, \langle\cdot,\cdot\rangle_Y$ denote the corresponding inner products and $\|\cdot\|_X, \|\cdot\|_Y$ the respective norms.

DEFINITION 3.2 We say X, Y form a compatible pair denoted by (X, Y) if X is a dense subset of Y and the injection $i: X \to Y$ is continuous, i.e., there exists $c > 0$ such that

$$\|v\|_Y = \|iv\|_Y \leqslant c\|v\|_X.$$

By the Riesz representation theory (see, e.g., Riesz & Sz-Nagy, 1956) there exists an operator $\mathcal{J}: X \to Y$ which is positive and self-adjoint with respect to $\langle\cdot,\cdot\rangle_Y$ such that

$$\langle u, v\rangle_X = \langle u, \mathcal{J}v\rangle_Y. \tag{3.1}$$

Using the spectral decomposition of $\mathcal{J}$ we define the operator $\mathcal{E} = \mathcal{J}^{1/2}: X \to Y$, which in turn is positive self-adjoint. Moreover, the spectral decomposition of $\mathcal{E}$ can be used to define any real power of $\mathcal{E}$. Let $\theta \in [0, 1]$ and let $\|\cdot\|_\theta$ denote the scale of graph norms

$$\|u\|\theta := (\|u\|Y^2 + \|\mathcal{E}^{1-\theta}u\|Y^2)^{1/2}. \tag{3.2}$$

One can then show that the domain of $\mathcal{E}^{1-\theta}$ endowed with the inner product

$$\langle u, v\rangle_\theta = \langle u, v\rangle_Y + \left\langle \mathcal{E}^{1-\theta}u, \mathcal{E}^{1-\theta}v\right\rangle_Y$$

is a Hilbert space (Lions & Magenes, 1968). This is an *interpolation space of index θ for the pair* $[X, Y]$ and is denoted by $[X, Y]_\theta$

$$[X, Y]_\theta := D(\mathcal{E}^{1-\theta}), \quad 0 \leqslant \theta \leqslant 1.$$

Let $\mathcal{V}_h \subset X$ denote a subset of X with $\dim \mathcal{V}_h = n$. Let now $X_h = (\mathcal{V}_h, \|\cdot\|_X)$ and $Y_h = (\mathcal{V}_h, \|\cdot\|_Y)$ be finite-dimensional subspaces of X and Y, respectively, with the indicated induced norm topologies. It follows that (X_h, Y_h) is a compatible pair of Hilbert spaces. We can similarly define corresponding positive self-adjoint operators $\mathcal{J}_h, \mathcal{E}_h : X_h \to Y_h$,

$$\langle u_h, v_h\rangle_X = \langle u_h, \mathcal{J}_h v_h\rangle_Y, \quad u_h, v_h \in X_h, \tag{3.3}$$

where $\mathcal{J}_h$ is positive self-adjoint and $\mathcal{E}_h = \mathcal{J}_h^{1/2}$. We define the discrete interpolation spaces

$$[X_h, Y_h]_\theta := D\big(\mathcal{E}_h^{1-\theta}\big).$$

Furthermore, we define the scale of discrete norms

$$\|u_h\|_{\theta,h} := (\|u_h\|_Y^2 + \|\mathcal{E}_h^{1-\theta} u_h\|_Y^2)^{1/2}. \tag{3.4}$$

The matrix representation of $\|\cdot\|_{\theta,h}$ with respect to some basis is given by Arioli & Loghin (2009),

$$H_\theta = H_Y + H_Y (H_Y^{-1} H_X)^{1-\theta}, \tag{3.5}$$

where H_X, H_Y are the grammians of the basis functions in the inner products of X, Y.

The following result can also be found in Arioli & Loghin (2009).

LEMMA 3.3 Let (X, Y), (X_h, Y_h) be compatible pairs of Hilbert spaces. Let $\|\cdot\|_\theta$, $\|\cdot\|_{\theta,h}$ denote norms on $[X, Y]_\theta$, $[X_h, Y_h]_\theta$, respectively. Let $I_h \in \mathcal{L}(X; X_h) \cap \mathcal{L}(Y; Y_h)$ satisfy

$$\|I_h u\|_{X_h} \leqslant M_0 \|u\|_X \quad (\forall\, u \in X), \qquad \|I_h u\|_{Y_h} \leqslant M_1 \|u\|_Y \quad (\forall\, u \in Y), \tag{3.6}$$

with constants M_0, M_1 independent of n. Assume further that $I_h u_h = u_h$ for all $u_h \in X_h$. Then $\|\cdot\|_\theta$, $\|\cdot\|_{\theta,h}$ are equivalent on $[X_h, Y_h]_\theta$ for all $\theta \in (0, 1)$ with constants independent of n.

3.2 *Interpolation on interfaces*

In this section we describe some assumptions that allow the approximation of functions and interpolation spaces by a finite element method on the set of internal boundaries.

In the following we assume that the domain decomposition is obtained by slicing Ω with a set $\{\mathfrak{m}_k, k = 1, \ldots, K\}$ of bounded Riemannian manifolds of dimension $(d-1)$, nontangent to each other, and such that the intersection of two of them is either $\emptyset$ or it is a regular manifold if $d > 2$. If $d = 2$, then each manifold is a curve and the possible intersection is a point. It follows that $\Gamma = \cup_k \mathfrak{m}_k$. Define $Y_k = L^2(\mathfrak{m}_k)$ and $Y = \cup_k Y_k = L^2(\Gamma)$. Let ∇_Γ^k denote the tangential gradient of v with respect to $\mathfrak{m}_k$,

$$\nabla_\Gamma^k v(\mathbf{x}) := \nabla v(\mathbf{x}) - \mathbf{n}_k(\mathbf{x})(\mathbf{n}_k(\mathbf{x}) \cdot \nabla v(\mathbf{x})), \tag{3.7}$$

where $\mathbf{n}_k(\mathbf{x})$ is the normal to $\mathfrak{m}_k$ at $\mathbf{x}$. In other words, $\nabla_\Gamma^k v$ is the projection of the gradient of v onto the hyperplane tangent to $\mathfrak{m}_k$ at $\mathbf{x}$. Finally, let $\partial\Gamma = \cup_k \partial\mathfrak{m}_k \cap \partial\Omega$ denote the set of points (faces) on the boundary of our domain which represent the skeleton boundary. We do not require that all the manifolds have $\partial\mathfrak{m}_k \cap \partial\Omega \neq \emptyset$ but we assume that $\partial\Gamma \neq \emptyset$. In the following we will use the notation introduced in Bertoluzza & Kunoth (2000),

$$H^1_{\partial\Omega}(\mathfrak{m}_k) = \begin{cases} H^1(\mathfrak{m}_k) := \left\{ v \in L^2(\mathfrak{m}_k) \colon \displaystyle\int_{\mathfrak{m}_k} |\nabla_\Gamma^k v|^2 \, \mathrm{d}s(\mathfrak{m}_k) < \infty \right\}, & \partial\mathfrak{m}_k \cap \partial\Omega = \emptyset, \\ H^1_0(\mathfrak{m}_k) := \{ v \in H^1(\mathfrak{m}_k) \colon v \,|_{\partial\mathfrak{m}_k} = 0 \}, & \partial\mathfrak{m}_k \cap \partial\Omega \neq \emptyset. \end{cases} \tag{3.8}$$

Under the previous assumptions, $H^1(\mathfrak{m}_k)$ is the closure of $\mathcal{C}^\infty(\mathfrak{m}_k)$ with the norm

$$\|v\|^2 H^1_{\partial\Omega}(\mathfrak{m}_k) = \int_{\mathfrak{m}_k} |v|^2 \, \mathrm{d}s(\mathfrak{m}_k) + \int_{\mathfrak{m}_k} |\nabla_\Gamma^k v|^2 \, \mathrm{d}s(\mathfrak{m}_k).$$

Finally,

$$H_0^1(\mathfrak{m}_k) := \{v \in H^1(\mathfrak{m}_k) \colon v\ |_{\partial \mathfrak{m}_k} = 0\}$$

is the closure of $\mathcal{C}_0^\infty(\mathfrak{m}_k)$ with

$$|v|^2_{H_0^1(\mathfrak{m}_k)} = \int_{\mathfrak{m}_k} |\nabla_\Gamma^k v|^2 \, \mathrm{d}s(\mathfrak{m}_k).$$

A function u on Γ will be defined by a K-tuple $(u_1, \ldots, u_K) \in \prod_{k=1}^K H^1_{\partial\Omega}(\mathfrak{m}_k)$, such that

$$u\ |_{\mathfrak{m}_k} = u_k.$$

We endow $\prod_{k=1}^K H^1_{\partial\Omega}(\mathfrak{m}_k)$ with the norm

$$\|u\|_{H^1_{\partial\Omega}(\Gamma)} = \left(\sum_{k=1}^K \|u_k\|^2_{H^1_{\partial\Omega}(\mathfrak{m}_k)} \right)^{1/2}.$$

Each u_k has a well-defined trace on the intersections with the other manifolds. In the following we will work with the space

$$X = \left\{ u \in \prod_{k=1}^K H^1_{\partial\Omega}(\mathfrak{m}_k);\ \gamma_0(\mathfrak{m}_j \cap \mathfrak{m}_k)(u_k) = \gamma_0(\mathfrak{m}_j \cap \mathfrak{m}_k)(u_j) \text{ if } \mathfrak{m}_j \cap \mathfrak{m}_k \neq \emptyset \right\}, \tag{3.9}$$

i.e., we glue together all the u_k at the intersections using the assumption that all the intersections are regular manifolds. X is a closed subspace of $\prod_{k=1}^K H^1_{\partial\Omega}(\mathfrak{m}_k)$ and dense in Y. Therefore, we can define the interpolation space between X and Y and we have that $[X, Y]_{1/2}$ is isometric to a subspace of $\prod_{k=1}^K H^{1/2}_{\partial\Omega}(\mathfrak{m}_k)$, with

$$H^{1/2}_{\partial\Omega}(\mathfrak{m}_k) = \begin{cases} H^{1/2}(\mathfrak{m}_k), & \partial\mathfrak{m}_k \cap \partial\Omega = \emptyset, \\ H^{1/2}_{00}(\mathfrak{m}_k), & \partial\mathfrak{m}_k \cap \partial\Omega \neq \emptyset. \end{cases}$$

The previous definitions are consistent with the framework introduced in Bertoluzza & Kunoth (2000).

REMARK 3.4 If $d = 2$, the intersections are points and our definition of X gives a space that is isometric to the one used in metric graph theory (Kuchment, 2003, 2008), where $X = H^1_{\partial\Omega}(\Gamma) \cap \mathcal{C}^0(\Gamma)$.

The space S^h introduced in Section 2.3 is the span of $\gamma_0(\Gamma)\psi_k$. These trace functions are continuous piecewise polynomial functions defined on the subdivision of $\mathfrak{m}_k$ that on the intersection of two manifolds have the same values. Let

$$S_{h,k} = S^h \cap \left\{ \xi_i = \gamma_0(\mathfrak{m}_k)\psi_i;\ 1 \leqslant k \leqslant n_B \right\}$$

and

$$X_{h,k} = \left(S_{h,k}, \|\cdot\|_{H^1_{\partial\Omega}(\mathfrak{m}_k)} \right), \quad Y_{h,k} = \left(S_{h,k}, \|\cdot\|_{L^2(\mathfrak{m}_k)} \right).$$

With this notation, a function in S^h is identified with a K-tuple of functions in $\prod_k S_{h,k}$ and S^h is isometric to

$$X_h := \left(\left\{ \xi \in \prod_k S_{h,k};\ \gamma_0(\mathfrak{m}_j \cap \mathfrak{m}_k)(\xi_k) = \gamma_0(\mathfrak{m}_j \cap \mathfrak{m}_k)(\xi_j) \text{ if } \mathfrak{m}_k \cap \mathfrak{m}_j \neq \emptyset \right\}, \| \cdot \|_{H^1_{\partial\Omega}(\Gamma)} \right).$$

Moreover, we can define

$$Y_h = \left(\prod_k Y_{h,k},\ \| \cdot \|_{L^2} \right).$$

X_h is then the span of continuous piecewise polynomial functions defined on the subdivision of $\mathfrak{m}_k$ that on the intersection of two manifolds have the same values.

Let now $(\cdot,\cdot)_X, (\cdot,\cdot)_Y$ denote the inner products that induce the corresponding norms on X, Y. Let M, L denote the norm representations of $\| \cdot \|_{L^2(\Gamma)}, \| \cdot \|_{H^1_{\partial\Omega}(\Gamma)}$ with respect to the basis $\{\gamma_0(\mathfrak{m}_k)\psi_i\}_{1 \leqslant i \leqslant n_B}$, i.e.,

$$M_{ij} = \sum_{k=1}^{K} (\gamma_0(\mathfrak{m}_k)\psi_i, \gamma_0(\mathfrak{m}_k)\psi_j)_{L^2(\mathfrak{m}_k)}, \quad L_{ij} = \sum_{k=1}^{K} (\gamma_0(\mathfrak{m}_k)\nabla^k_\Gamma \psi_i, \gamma_0(\mathfrak{m}_k)\nabla^k_\Gamma \psi_j)_{H^1_{\partial\Omega}(\mathfrak{m}_k)}$$

for $i, j = 1, \ldots, n_B$. A norm for the discrete interpolation space $[X_h, Y_h]_{1/2}$ has the matrix representation (cf. (3.5))

$$H_{1/2} = M + M(M^{-1}L)^{1/2}, \tag{3.10}$$

which can be shown to be spectrally equivalent to (Arioli & Loghin, 2009)

$$\widehat{H}_{1/2} = M(M^{-1}L)^{1/2}. \tag{3.11}$$

REMARK 3.5 The matrix L can be seen as a discrete Laplace–Beltrami operator assembled on Γ. For the case when $\mathfrak{m}_k$ are $(d-1)$-dimensional hyperplanes, the matrix L has the sparsity of a $(d-1)$-dimensional Laplacian matrix with d-dimensional subblocks corresponding to the subset $\cap_k \mathfrak{m}_k$. For example, for a cross in two dimensions, a piecewise linear discretization yields a matrix L which is tridiagonal except for a five-point stencil corresponding to the cross-point (vertex and four neighbouring nodes).

With this notation, Lemma 3.3 applies to the pairs $(X_{h,k}, Y_{h,k})$ with I_h the piecewise polynomial interpolation operator associated with the nodes in the subdivision of $\mathfrak{m}_k$. In particular, on each interpolation space $[X_{h,k}, Y_{h,k}]_{1/2}$ we have an equivalence between the continuous and discrete interpolation norms of index 1/2. Summing over k, we obtain an equivalence on $[X_h, Y_h]_{1/2}$, namely, for all $\lambda_h \in S^h$ with $\lambda_h = \sum_{i=1}^{n_B} \boldsymbol{\lambda}_i \psi_i$ there exist constants κ_1, κ_2 such that

$$\kappa_1 \|\lambda_h\|_{1/2,\Gamma} \leqslant \|\boldsymbol{\lambda}\|_{H_{1/2}} \leqslant \kappa_2 \|\lambda_h\|_{1/2,\Gamma}. \tag{3.12}$$

We immediately derive the following result.

PROPOSITION 3.6 Let $s(\cdot,\cdot)$ be defined as in (2.16) and let (2.6) hold. Let $\boldsymbol{\eta}, \boldsymbol{\mu}$ denote the coefficients of η_h, μ_h with respect to the basis $\{\psi_i, i = 1, \ldots, n_B\}$ of S^h. Let S denote the matrix representation of $s(\cdot,\cdot)$ with respect to the same basis. Then there exist constants $\tilde{\alpha}_1, \tilde{\alpha}_2, \widehat{\alpha}_1, \widehat{\alpha}_2$ such that

$$\tilde{\alpha}_1 \|\boldsymbol{\eta}\|^2_{H_{1/2}} \leqslant \boldsymbol{\eta}^{\mathrm{T}} S \boldsymbol{\eta}, \quad \boldsymbol{\mu}^{\mathrm{T}} S \boldsymbol{\eta} \leqslant \tilde{\alpha}_2 \|\boldsymbol{\eta}\|_{H_{1/2}} \|\boldsymbol{\mu}\|_{H_{1/2}}$$

and

$$\widehat{\alpha}_1 \|\boldsymbol{\eta}\|^2_{\widehat{H}_{1/2}} \leqslant \boldsymbol{\eta}^{\mathrm{T}} S \boldsymbol{\eta}, \quad \boldsymbol{\mu}^{\mathrm{T}} S \boldsymbol{\eta} \leqslant \widehat{\alpha}_2 \|\boldsymbol{\eta}\|_{\widehat{H}_{1/2}} \|\boldsymbol{\mu}\|_{\widehat{H}_{1/2}}$$

for all $\eta_h, \mu_h \in S_h \subset H^{1/2}_{00}(\Gamma)$.

Proof. The results follow from Lemma 3.1 and equivalence (3.12). □

We will see below that the above equivalence indicates that the norm-matrices $H_{1/2}$, $\widehat{H}_{1/2}$ can be used as preconditioners for the Schur complement and that they are optimal in some sense to be described.

3.3 *Mesh-independent preconditioners*

The solution of the linear system requires an iterative approach in the case of large scale problems. A useful approach is to consider an iterative solver such as GMRES together with a suitable preconditioning strategy. In our case one could, for example, employ a right preconditioner that will incorporate the solution of problems posed on the interior of each domain (achieved in parallel) and the (approximate) solution of a problem involving the discrete Steklov–Poincaré operator. Given the equivalence in Proposition 3.6, a candidate as right preconditioner can be taken to be

$$P_R = \begin{pmatrix} A_{II} & A_{IB} \\ 0 & H_{1/2} \end{pmatrix}.$$

With this choice, the preconditioned system is

$$AP_R^{-1} = \begin{pmatrix} I & 0 \\ A_{BI}A_{II}^{-1} & SH_{1/2}^{-1} \end{pmatrix}.$$

This block structure indicates that the convergence of an iterative algorithm such as GMRES will depend on the ability of $H_{1/2}$ to approximate S. In particular, the eigenvalues of the above preconditioned matrix are either equal to 1 or coincide with one of the eigenvalues of $SH_{1/2}^{-1}$. Note that the spectrum of S depends on n_B and also on the subdomain decomposition: number of subdomains, partitioning configuration, subdomain regularity, etc. We show below that the eigenvalues of $SH_{1/2}^{-1}$ lie in a region of the complex plane which is independent of the size of the problem n_B and which also lies in the right half plane. We will investigate numerically the dependence on the type of decomposition employed.

We start by recalling the definition of the H-field of values of a matrix A, given a symmetric and positive-definite matrix H.

DEFINITION 3.7 Let $R, H \in \mathbb{R}^{n\times n}$, with H symmetric and positive definite. The H-field of values of the matrix R, denoted by $\mathcal{W}_H(R)$, is a set in the complex plane given by

$$\mathcal{W}_H(R) = \left\{ z \in \mathbb{C} : z = \frac{\mathbf{x}^* H R \mathbf{x}}{\mathbf{x}^* H \mathbf{x}} = \frac{\langle \mathbf{x}, R\mathbf{x}\rangle_H}{\langle \mathbf{x}, \mathbf{x}\rangle_H}, \mathbf{x} \in \mathbb{C}^n \setminus \{0\} \right\}.$$

When $H = I$, the set is called the field of values and is denoted by $\mathcal{W}(R)$.

We also need to recall a related result concerning the convergence of GMRES (see Elman, 1982; Saad, 1996).

LEMMA 3.8 Let $H \in \mathbb{R}^{n\times n}$ be a positive-definite matrix. Let $R, P \in \mathbb{R}^{n\times n}$ be nonsingular matrices such that the following bounds hold:

$$\xi_1 \leqslant \frac{\langle \mathbf{x}, RP^{-1}\mathbf{x}\rangle_H}{\langle \mathbf{x}, \mathbf{x}\rangle_H}, \quad \frac{\|RP^{-1}\mathbf{x}\|_H}{\|\mathbf{x}\|_H} \leqslant \xi_2 \tag{3.13}$$

for some positive constants ξ_1 and ξ_2. Then the GMRES algorithm in the H-inner product yields a residual $\mathbf{r}^k$ after k iterations which satisfies

$$\frac{\|\mathbf{r}^k\|_H}{\|\mathbf{r}^0\|_H} \leqslant \left(1 - \frac{\xi_1^2}{\xi_2^2}\right)^{k/2}. \tag{3.14}$$

The following result provides bounds on the $H_{1/2}^{-1}$-field of values.

PROPOSITION 3.9 Let the hypothesis of Proposition 3.6 hold. Then the $H_{1/2}^{-1}$-field of values of $SH_{1/2}^{-1}$ is in the right half plane and is bounded independently of n_B.

Proof. The projection on the real line of the $H_{1/2}^{-1}$-field of values is bounded from below by

$$\min_{z\in\mathcal{W}_{H_{1/2}^{-1}}(SH_{1/2}^{-1})} |z| = \min_{\boldsymbol{\eta}\in\mathbb{R}^{n_B}\setminus\{\mathbf{0}\}} \frac{\langle \boldsymbol{\eta} SH_{1/2}^{-1}\boldsymbol{\eta}\rangle_{H_{1/2}^{-1}}}{\langle \boldsymbol{\eta}, \boldsymbol{\eta}\rangle_{H_{1/2}^{-1}}} = \min_{\boldsymbol{\eta}\in\mathbb{R}^{n_B}\setminus\{\mathbf{0}\}} \frac{\boldsymbol{\eta}^{\mathrm{T}} S\boldsymbol{\eta}}{\boldsymbol{\eta}^{\mathrm{T}} H_{1/2}\boldsymbol{\eta}} \geqslant \tilde{\alpha}_1 > 0.$$

An upper bound for the field of values is provided by the numerical radius which in turn is bounded by the maximum $H_{1/2}^{-1}$-singular value. The resulting bound on the $H_{1/2}^{-1}$-field of values of $SH_{1/2}^{-1}$ is

$$|z| \leqslant \max_{\boldsymbol{\eta}\in\mathbb{R}^{n_B}\setminus\{\mathbf{0}\}} \frac{\|S\boldsymbol{\eta}\|_{H_{1/2}^{-1}}}{\|\boldsymbol{\eta}\|_{H_{1/2}}} = \max_{\boldsymbol{\eta}\in\mathbb{R}^{n_B}\setminus\{\mathbf{0}\}} \max_{\boldsymbol{\mu}\in\mathbb{R}^{n_B}\setminus\{\mathbf{0}\}} \frac{\boldsymbol{\eta} S\boldsymbol{\mu}}{\|\boldsymbol{\eta}\|_{H_{1/2}}\|\boldsymbol{\mu}\|_{H_{1/2}}} \leqslant \tilde{\alpha}_2.$$

□

Note that the above bounds also imply the following bounds independent of n_B on the eigenvalues of the preconditioned discrete Steklov–Poincaré operator:

$$\tilde{\alpha}_1 \leqslant |\lambda(SH_{1/2}^{-1})| \leqslant \tilde{\alpha}_2.$$

Given the result of Proposition 3.6, a convergence bound can be immediately derived for a system of equations with $SH_{1/2}^{-1}$ as a coefficient matrix.

PROPOSITION 3.10 Let the hypothesis of Proposition 3.6 hold. Then the GMRES algorithm applied to the linear system

$$SH_{1/2}^{-1}\tilde{\mathbf{y}} = \mathbf{z}, \quad (\tilde{\mathbf{y}} = H_{1/2}\mathbf{y})$$

in the $H_{1/2}^{-1}$-inner product yields a residual $\mathbf{r}^k$ after k iterations which satisfies

$$\frac{\|\mathbf{r}^k\|_{H_{1/2}^{-1}}}{\|\mathbf{r}^0\|_{H_{1/2}^{-1}}} \leqslant \left(1 - \frac{\tilde{\alpha}_1^2}{\tilde{\alpha}_2^2}\right)^{k/2}. \tag{3.15}$$

The following result is adapted from Loghin & Wathen (2004, Theorem 3.7).

PROPOSITION 3.11 Let the hypothesis of Proposition 3.6 hold and let P_R be given by

$$P_R = \begin{pmatrix} A_{II} & A_{IB} \\ 0 & \rho H_{1/2} \end{pmatrix}.$$

Then there exists $\rho_0 > 0$ such that for all $\rho > \rho_0$ conditions (3.13) hold with R, P replaced by A, P_R and for the choice

$$H = \begin{pmatrix} A_{II} & 0 \\ 0 & H_{1/2} \end{pmatrix}^{-1}.$$

As before, this result indicates that block triangular preconditioners $P_R(\rho)$ are optimal preconditioners when we use a suitable GMRES iteration to obtain the solution of the *global linear system*.

REMARK 3.12 All results in this section apply with $H_{1/2}$ replaced by $\widehat{H}_{1/2}$ as defined in (3.11).

3.4 *Algorithms for the matrix square root*

The discrete fractional Sobolev norms introduced in Section 3.3 require the evaluation of the square root function of a matrix of size $n_B = \mathcal{O}(n^{1-1/d})$. According to application this task can be achieved in different ways. Direct approaches are based on a generalised eigenvalue decomposition that is known to have complexity $\mathcal{O}(n_B^3)$. For problems with structure we can also employ fast algorithms such as the FFT (Peisker, 1988) with reduced complexity ($\mathcal{O}(n_B \log n_B)$). The alternative is to use iterative techniques. Standard iterations, such as Newton's method, may not have better complexity than a direct method. Others could claim reduced complexity (Hale *et al.*, 0000). In our implementation we used a Krylov subspace approximation that takes advantage of the sparsity properties of the matrices involved in the definition of our discrete fractional Sobolev norms. In particular, we employed a generalized Lanczos algorithm that we describe below together with some related approximations required inside a preconditioning procedure.

3.5 *Generalized Lanczos algorithms*

Given a pair of symmetric and positive-definite matrices $L, M \in \mathbb{R}^{n_B \times n_B}$, the generalized Lanczos algorithm constructs a set of M-orthogonal vectors $\mathbf{v}_i \in \mathbb{R}^{n_B}$ such that

$$LV_k = MV_kT_k + \beta_{k+1}M\mathbf{v}_{k+1}\mathbf{e}_k^{\mathrm{T}}, \quad V_k^{\mathrm{T}}MV_k = I_k,$$

where the columns $\mathbf{v}_i$ of $V_k = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k]$ are known as the Lanczos vectors and $I_k \in \mathbb{R}^{k\times k}$ is the identity matrix with kth column denoted by $\mathbf{e}_k$, while the matrix $T_k \in \mathbb{R}^{k\times k}$ is a symmetric and tridiagonal matrix (Parlett, 1998). The standard algorithm corresponds to the case $M = I$. Note that T_k can be seen as a projection of L onto the space spanned by the M-orthogonal columns of V_k,

$$V_k^{\mathrm{T}}LV_k = T_k, \quad V_k^{\mathrm{T}}MV_k = I_k. \tag{3.16}$$

In exact arithmetic, when $k = n_B$, the algorithm can be seen as providing simultaneous factorizations of the matrix pair (M, L) as

$$L = V_n^{-\mathrm{T}}T_nV_n^{-1}, \quad M = V_n^{-\mathrm{T}}V_n^{-1}.$$

The above procedure is typically employed for the computation of the eigenvalues of a symmetric and positive-definite matrix; the generalized version achieves this for the matrix $M^{-1}L$. In particular, the above factorizations can be written as a similarity between $M^{-1}L$ and the tridiagonal matrix T_n.

Another approach to computing the eigenvalues of the pair (L, M) is to consider the *inverse Lanczos algorithm* which is the Lanczos algorithm applied to the pair (M^{-1}, L^{-1}). While this appears to be computationally more expensive, the convergence of the smallest eigenvalues of T_k to those of $M^{-1}L$ may be more favourable. For the purpose of preconditioning this is a desirable feature. We illustrate this point in the numerics section.

Consider now the generalized Lanczos factorisation for the matrix pairs (M, L) and (L, M) written as

$$LV = MVT, \quad V^{\mathrm{T}}MV = I, \tag{3.17}$$

$$MW = LWR, \quad W^{\mathrm{T}}LW = I, \tag{3.18}$$

where we used the notation $V = V_{n_B}, T = T_{n_B}$ and $W = W_{n_B}, R = R_{n_B}$ with T, R tridiagonal matrices. We can immediately derive the following result.

LEMMA 3.13 Let (3.17), (3.18) hold and let

$$H_{1/2} = M + M(M^{-1}L)^{1/2}, \quad \widehat{H}_{1/2} = M(M^{-1}L)^{1/2}.$$

Then

$$H_{1/2}^{-1} = V(I + T^{1/2})^{-1}V^{\mathrm{T}} = W(I + R^{1/2})^{-1}W^{\mathrm{T}}$$

and

$$\widehat{H}_{1/2}^{-1} = VT^{-1/2}V^{\mathrm{T}} = WR^{-1/2}W^{\mathrm{T}}.$$

The complexity of the full ($k = n_B$) generalized Lanczos algorithm is in general $\mathcal{O}(n_B^3)$. However, in many applications of interest, we do not need to compute $H_{1/2}$, but simply apply it (or its inverse) to a given vector $\mathbf{z} \in \mathbb{R}^n$. In such cases, a truncated version of the algorithm is used in practice with only k Lanczos vectors being constructed. As we are interested in approximations of $H_{1/2}^{-1}\mathbf{z}$ we note first that if we start the Lanczos process with $\mathbf{v}_1 = M^{-1}\mathbf{z}$ then

$$V_k^{\mathrm{T}}\mathbf{z} = V_k^{\mathrm{T}}M(M^{-1}\mathbf{z}) = \mathbf{e}_1\|M^{-1}\mathbf{z}\|_M = \mathbf{e}_1\|\mathbf{z}\|_{M^{-1}},$$

where $\mathbf{e}_1 \in \mathbb{R}^k$ is the first column of the identity matrix I_k. Similarly, if we start the inverse Lanczos process with $\mathbf{w}_1 = L^{-1}\mathbf{z}$, then

$$W_k^{\mathrm{T}}\mathbf{z} = W_k^{\mathrm{T}}L(L^{-1}\mathbf{z}) = \mathbf{e}_1\|L^{-1}\mathbf{z}\|_L = \mathbf{e}_1\|\mathbf{z}\|_{L^{-1}}.$$

This leads us to consider the following approximations (cf. Lemma 3.13):

$$H_{1/2}^{-1}\mathbf{z} \approx V_k\big(I_k + T_k^{1/2}\big)^{-1}V_k^{\mathrm{T}}\mathbf{z} = V_k\big(I_k + T_k^{1/2}\big)^{-1}\mathbf{e}_1\|\mathbf{z}\|_{M^{-1}}, \tag{3.19}$$

$$H_{1/2}^{-1}\mathbf{z} \approx W_k\big(I_k + R_k^{1/2}\big)^{-1}W_k^{\mathrm{T}}\mathbf{z} = W_k\big(I_k + R_k^{1/2}\big)^{-1}\mathbf{e}_1\|\mathbf{z}\|_{L^{-1}}. \tag{3.20}$$

Similar expressions can be obtained for the product $\widehat{H}_{1/2}^{-1}\mathbf{z}$:

$$\widehat{H}_{1/2}^{-1}\mathbf{z} \approx V_k T_k^{-1/2} V_k^{\mathrm{T}}\mathbf{z} = V_k T_k^{-1/2}\mathbf{e}_1\|\mathbf{z}\|_{M^{-1}} \tag{3.21}$$

$$\widehat{H}_{1/2}^{-1}\mathbf{z} \approx W R_k^{-1/2} W_k^{\mathrm{T}}\mathbf{z} = W_k R_k^{-1/2}\mathbf{e}_1\|\mathbf{z}\|_{L^{-1}}. \tag{3.22}$$

REMARK 3.14 The complexity of the above operations depends on the complexity corresponding to the application of M^{-1}, L^{-1}. This can be achieved in $\mathcal{O}(n_B)$ operations so that the overall complexity of computing $H_{1/2}\mathbf{z}$, $H_{1/2}^{-1}\mathbf{z}$ is of order $\mathcal{O}(kn_B)$ for $k \ll n_B$, with storage requirements of the same order. Hence, the complexity of applying the boundary preconditioner is $\mathcal{O}(n^{1-1/d})$, where n is the size of the global problem, which for most practical values of N is far below that of solving a subdomain exactly (order $\mathcal{O}((n/N)^3)$).

4. Numerical experiments

We performed a series of experiments on standard elliptic problems. The domains were subdivided *a priori* (prior to triangulation) so that the resulting subdomain boundaries were linear (planar) faces. This allows for tangential gradient (3.7) to be implemented in a natural way. In both cases we used a number of levels of refinement to investigate performance. Mesh information is included for each test problem. We also present results corresponding to subdividing the computational domain *a posteriori* (after triangulation) using the automatic partitioning tool METIS (Karypis & Kumar, 1998, 1999). For this case we also used several levels of refinement.

The iterative method employed in all cases is the GMRES method with right preconditioners,

$$P_R = \begin{pmatrix} A_{II} & A_{IB} \\ 0 & H_{1/2} \end{pmatrix}, \quad \widehat{P}_R = \begin{pmatrix} A_{II} & A_{IB} \\ 0 & \widehat{H}_{1/2} \end{pmatrix},$$

and we recall here that

$$H_{1/2} = M + M(M^{-1}L)^{1/2}, \quad \widehat{H}_{1/2} = M(M^{-1}L)^{1/2}.$$

On a uniform mesh with mesh size h, the mass matrix is known to be spectrally equivalent to its diagonal: $\widetilde{M}$. This yields a more practical approximation $\widehat{H}_{1/2}$,

$$\widetilde{H}_{1/2} = \widetilde{M}(\widetilde{M}^{-1}L)^{1/2}.$$

We denote the corresponding block-triangular preconditioner by $\widetilde{P}_R$.

The preconditioners were implemented using the following decomposition of the inverse preconditioner:

$$P_R^{-1} = \begin{pmatrix} A_{II}^{-1} & 0 \\ 0 & I_{BB} \end{pmatrix}\begin{pmatrix} I_{II} & -A_{IB} \\ 0 & I_{BB} \end{pmatrix}\begin{pmatrix} I_{II} & 0 \\ 0 & H_{1/2}^{-1} \end{pmatrix}.$$

In our implementation both direct and iterative methods are exploited. The direct method computes the generalized eigenvalue decomposition, while the iterative method is based on Lanczos approximations (3.19), (3.21), in which case we have used the flexible GMRES method (Saad, 1993) to take into account

the changing nature of the preconditioner. Note that while in some cases the problem is symmetric, our preconditioner is nonsymmetric—we found that the block-triangular preconditioner introduced in the previous section outperformed standard symmetric Krylov solvers. Not least, our aim was to monitor methodically the performance of a single (nonsymmetric) iterative method as the problems become more and more nonsymmetric (e.g., the convection–diffusion problem with diminishing diffusion). The direct sparse linear solver PARDISO (Schenk & Gärtner, 2002, 2006) was used for the local linear systems corresponding to each subdomain. Finally, we remark here that we did not use a two-level approach in order to highlight the raw performance of our preconditioners and in particular the independence of the mesh parameter h. Clearly, a multilevel approach can also be considered in this case in a standard fashion.

4.1 *Two-dimensional results*

In this section we present the numerical experiments obtained by solving some standard elliptic problems in two dimensions. The problems were solved on the same domain $\Omega = (-1, 1)^2$. We decomposed Ω into $N = N_x \times N_y$ subdomains of size $2/N_x \times 2/N_y$ each, with $N_x = N_y \in \{2, 4, 8, 16\}$. For the purpose of investigating performance, each subdomain was triangulated uniformly so that we work with a sequence of nested grids as well as nested subdomain partitions. The mesh/subdomain information is shown in Table 1 together with the number of nodes n_B on the internal boundary (skeleton). We also used the same domain decomposition with nonuniform mesh refinements with parameters n, n_B of the same order as those corresponding to the uniform case. We chose not to include the mesh information for the nonuniform case in order to keep the presentation of numerical results succinct.

PROBLEM 4.1 Consider the following standard model problem:

$$\begin{cases} -\mathrm{div}(\mathbf{a}\nabla u) = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \Omega. \end{cases} \tag{4.1}$$

The results for the choice $\mathbf{a} = \mathbf{1}$ and with preconditioner $\widehat{P}_R$ implemented exactly are shown in Table 2. We find indeed that the number of iterations is independent of the size of the problem n and is low but

TABLE 1 *Mesh information for two-dimensional experiments*

		n_B			
Level	n	$N = 4$	$N = 16$	$N = 64$	$N = 256$
1	16,641	253	753	1,729	3,585
2	66,049	509	1,521	3,521	7,425
3	263,169	1,021	3,057	7,105	15,105

TABLE 2 *GMRES iterations for Problem* 1 *using a uniform mesh*

Preconditioner	$\widehat{P}_R$				$\widetilde{P}_R$			
Domains	4	16	64	256	4	16	64	256
Level = 1	12	16	19	24	10	13	17	21
2	13	17	20	25	11	14	18	22
3	13	18	20	26	11	15	18	22

increases sublinearly as the size of the subdomains is reduced. We also show in Table 2 that the results for the simplified choice $\widetilde{P}_R$ computed exactly. As in the first case the number of iterations is independent of the size of the problem n but exhibits a sublinear dependence on the number of subdomains, though the number of iterations is slightly reduced.

Similar results are obtained using a sequence of nonnested nonuniform meshes. The results are shown in Table 3 and they indicate similar convergence behaviour: independence of level (size of problem) and sublinear dependence on the number of subdomains.

More realistic problems correspond to the case where the diffusion tensor $\mathbf{a}$ is not constant. In this case one can use the same interface preconditioner arising from the discrete interpolation pair $(L, M)_{1/2}$; however, the performance in this case will depend on the variation of $\mathbf{a}$ over the mesh. A more suitable interpolation pair employs a weighted Laplacian matrix, with weight equal to the trace of $\mathbf{a}$ on Γ. We denote this matrix by $L_{\mathrm{tr}\,\mathbf{a}}$; the corresponding interpolation pair is $(L_{\mathrm{tr}\,\mathbf{a}}, M)_{1/2}$. The results for both choices of preconditioner for a given scalar nonconstant $\mathbf{a}$ are shown in Table 4.

Finally, we investigate the performance of the interpolation pair $(L_{\mathrm{tr}\,\mathbf{a}}, M)_{1/2}$ for the case where $\mathbf{a}$ is a scalar function with a jump of magnitude μ across a fixed horizontal line, $\mathbf{a} = a(x, y)I_2$, where

$$a(x, y) = \begin{cases} \mu, & -1 < y < 1/4, \\ 1/\mu, & 1/4 \leqslant y < 1. \end{cases}$$

It is evident that the weighted interpolation pair yields a robust preconditioner with respect to the jumps in the diffusion coefficient. An analysis and generalizations to the full tensor case will be performed elsewhere.

PROBLEM 4.2 We consider now the following model for reaction–diffusion problems posed on the same domain and using the discretizations detailed in Table 1.

$$\begin{cases} -\varepsilon \Delta u + cu = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

TABLE 3 *GMRES iterations for Problem* 1 *using a nonuniform mesh*

Preconditioner	$\widehat{P}_R$				$\widetilde{P}_R$			
Domains	4	16	64	256	4	16	64	256
Level = 1	12	15	19	27	9	12	16	23
2	12	15	20	27	9	12	16	23
3	13	16	20	26	10	12	16	23

TABLE 4 *GMRES iterations for Problem* 1: $\mathbf{a} = (1 + 4x^2 + 4y^2 - 2xy)I_2$

Preconditioner	$(L_{\mathrm{tr}\,\mathbf{a}}, M)_{1/2}$				$(L, M)_{1/2}$			
Domains	4	16	64	256	4	16	64	256
Level = 1	13	16	19	25	26	33	44	59
2	14	16	19	26	30	37	46	62
3	14	16	20	26	33	40	48	62

It is known that decreasing ε is supposed to make the problem 'easier' to solve iteratively due to a increasingly dominant mass matrix. However, our problem develops boundary layers with decreasing ε which may lead to unstable layers near the boundary. Our preconditioning strategy reflects this behaviour for the case of the interpolation pair $(L, M)_{1/2}$. The iteration counts for $\varepsilon = 10^{-1}$, $\varepsilon = 10^{-2}$ and $\varepsilon = 10^{-3}$ are shown in Table 6. A clear improvement and independence of ε and of the number of subdomains is obtained for the interpolation pair $(\varepsilon L + M, M)$. The results for this case are shown in Table 7. Aside from independence of problem size, we see that a smaller ε leads to independence of number of domains as well.

A more realistic test for the reaction–diffusion problem corresponds to the case when c is not constant. It becomes more evident in this case that choosing a suitable interpolation pair is essential for the efficiency of the problem. Results for the case $c(x, y) = 1 + 4x^2 + 4y^2$ are included in Tables 8 and 9 for the choice of interpolation pairs $(\varepsilon L + M_{\mathrm{tr}\, c}, M)$ and $(\varepsilon L, M)$, respectively; here, we denoted by $M_{\mathrm{tr}\, c}$ the weighted mass matrix assembled on the interface Γ with weight equal to the trace $\mathrm{tr}\, c$ on Γ of $c(x, y)$. We see for these choices of preconditioners that the performance can deteriorate, instead of improve, with ε. This fact does not disprove the theoretical bounds derived in Section 4 but simply highlights the fact that the bounds do not describe performance when other parameters in the problem vary. In particular, we note that the results in Table 9 while still indicating mesh independence for $\varepsilon = 10^{-2}$,

TABLE 5 *GMRES iterations for Problem* 1 *using* $(L_{tr\mathbf{a}}, M)_{1/2}$

Level	1			2			3		
Domains	4	16	64	4	16	64	4	16	64
$\mu = 1$	13	16	18	14	17	19	14	17	20
10	15	18	22	15	18	23	16	19	24
100	15	18	22	15	18	23	16	19	25
1000	15	18	22	15	18	23	16	19	25

TABLE 6 *GMRES iterations for Problem* 2 *with* $c = 1$ *using the interpolation pair* $(L, M)_{1/2}$

ε	10^{-1}			10^{-2}			10^{-3}		
Domains	4	16	64	4	16	64	4	16	64
Level = 1	11	14	15	11	14	14	12	18	18
2	11	15	16	12	15	15	15	19	20
3	12	16	18	12	15	17	16	21	23

TABLE 7 *GMRES iterations for Problem* 2 *with* $c = 1$ *using the pair* $(\varepsilon L + M, M)_{1/2}$

ε	10^{-1}			10^{-2}			10^{-3}		
Domains	4	16	64	4	16	64	4	16	64
Level = 1	12	15	18	11	13	15	11	12	12
2	12	16	19	11	13	16	11	12	12
3	12	17	20	11	14	17	11	12	12

TABLE 8 *GMRES iterations for Problem* 2 *with* $c(x, y) = 1 + 4x^2 + 4y^2$ *using* $(\varepsilon L + M_{\mathrm{tr}\,c}, M)_{1/2}$

ε	10^{-2}			10^{-3}			10^{-4}		
Domains	4	16	64	4	16	64	4	16	64
Level = 1	11	13	15	11	14	14	11	14	15
2	11	13	14	11	13	14	11	14	14
3	11	13	14	11	13	14	11	12	13

TABLE 9 *GMRES iterations for Problem* 2 *with* $c(x, y) = 1 + 4x^2 + 4y^2$ *using* $(\varepsilon L, M)_{1/2}$

ε	10^{-2}			10^{-3}			10^{-4}		
Domains	4	16	64	4	16	64	4	16	64
Level = 1	11	15	15	12	19	22	15	24	28
2	11	16	16	14	20	23	17	26	31
3	12	17	17	15	22	25	20	29	34

TABLE 10 *GMRES iterations for Problem* 3 *with 'diagonal wind' using* $\widetilde{P}_R$ *and a uniform mesh*

ν	1				10^{-1}				10^{-2}			
Domains	4	16	64	256	4	16	64	256	4	16	64	256
Level = 1	13	17	22	30	13	16	22	29	19	26	34	43
2	13	17	22	30	13	17	22	29	20	28	34	42
3	14	17	22	30	13	17	22	29	21	29	35	42

they are still in an asymptotic regime for the other two values of ε. This is explained by the fact that the condition number of the problem preconditioned with the interpolation pair $(\varepsilon L, M)$ is of order $\mathcal{O}(\varepsilon^{-1})$.

PROBLEM 4.3 Consider the following convection–diffusion problem:

$$\begin{cases} -\nu\,\Delta u + \vec{b}\cdot\nabla u = 0 & \text{in } \Omega, \\ u = u_0 & \text{on } \partial\Omega, \end{cases} \tag{4.3}$$

with 'diagonal wind' $\vec{b} = (-1, -1)$ and also with 'rotating wind' $\vec{b} = (2y(1 - x^2), -2x(1 - y^2))$. The boundary data u_0 was chosen to be constant on some part of the boundary and zero elsewhere.

This is a nonsymmetric problem with a nonsymmetric Steklov–Poincaré operator that results in a nonsymmetric boundary Schur complement. While our preconditioners are nonsymmetric, the matrices $H_{1/2}$, $\widehat{H}_{1/2}$ and $\widetilde{H}_{1/2}$ are all symmetric and positive definite. We would therefore expect performance to deteriorate as the degree of nonsymmetry increases, which is the case for decreasing ν. We solved the problem using a standard stabilization technique: streamline upwinding Petrov–Galerkin. Our implementation adds an amount of diffusion in the direction of $\vec{b}$ which decreases with the Péclet number $\mathrm{Pe} = h\|\vec{b}\|/\nu$.

The results are shown in Table 10 for diagonal wind and Table 11 for rotating wind. In both cases we see that independence of the size of the problem holds also for this test problem, as predicted by theory. Finally, the sublinear dependence on the number of subdomains remains unchanged.

TABLE 11 *GMRES iterations for Problem* 3 *with 'rotating wind' using* $\widetilde{P}_R$ *and a uniform mesh*

ν	1				10^{-1}				10^{-2}			
Domains	4	16	64	256	4	16	64	256	4	16	64	256
Level = 1	13	17	22	30	11	18	27	37	22	38	60	86
2	13	17	22	31	12	18	27	37	21	37	58	82
3	13	17	23	31	12	18	26	37	20	37	55	75

TABLE 12 *Mesh information for three-dimensional experiments*

	n			n_B		
Domains	8	64	512	8	64	512
Level = 1	28,603	29,943	34,821	3,214	9,231	20,579
2	229,041	238,839	255,606	12,880	38,980	89,677
3	1,884,996	1,902,206	1,939,420	53,460	158,733	364,470

TABLE 13 *FGMRES iterations for Problem* 1 *(three-dimensional) approximating* $\widetilde{P}_R$ *with Lanczos:* $k = 3, 20$

Lanczos approx	$k = 3$			$k = 20$		
Domains	8	64	512	8	64	512
Level = 1	19	26	39	18	26	39
2	22	25	38	19	26	37
3	26	28	32	20	27	38

4.2 *Three-dimensional results*

We now present the numerical experiments obtained by solving the three-dimensional version of problem (4.1) posed in $\Omega = (0, 1)^3$. We decomposed Ω into $N = N_x \times N_y \times N_z$ subdomains of size $1/N_x \times 1/N_y \times 1/N_z$ each, with $N_x = N_y = N_z \in \{2, 4, 8\}$. The mesh generator employed was TetGen (TetGen, 2007; Si, 2008). Each subdomain contains approximately the same number of tetrahedra. The mesh/subdomain information is shown in Table 12 together with the number of nodes n_B on the skeleton.

The results are shown in Table 13. The boundary preconditioner used in our block-triangular preconditioners was approximated using the Lanczos approximations of Section 4 with $k = 20$. Note that this choice is not optimal for all mesh configurations—see Section 4.3 for details. As before, we see a domain dependence which is sublinear and virtually no dependence on the size of the problem. However, the latter property is somewhat affected when the Lanczos approximation is poor.

We end with results corresponding to an exotic choice of domain for Problem 1: the crystal depicted in Fig. 1 together with a METIS decomposition. The relevance of this problem consists mainly in the discretization employed: the mesh is highly anisotropic with large variations in the mesh size (up to 10^5) on each level of discretization. This type of problem illustrates the need for maintaining the mass matrix in the preconditioner leading to suitably scaled preconditioners with performance independent of mesh variations. The numerical results for the direct Lanczos method are included in Table 14 for

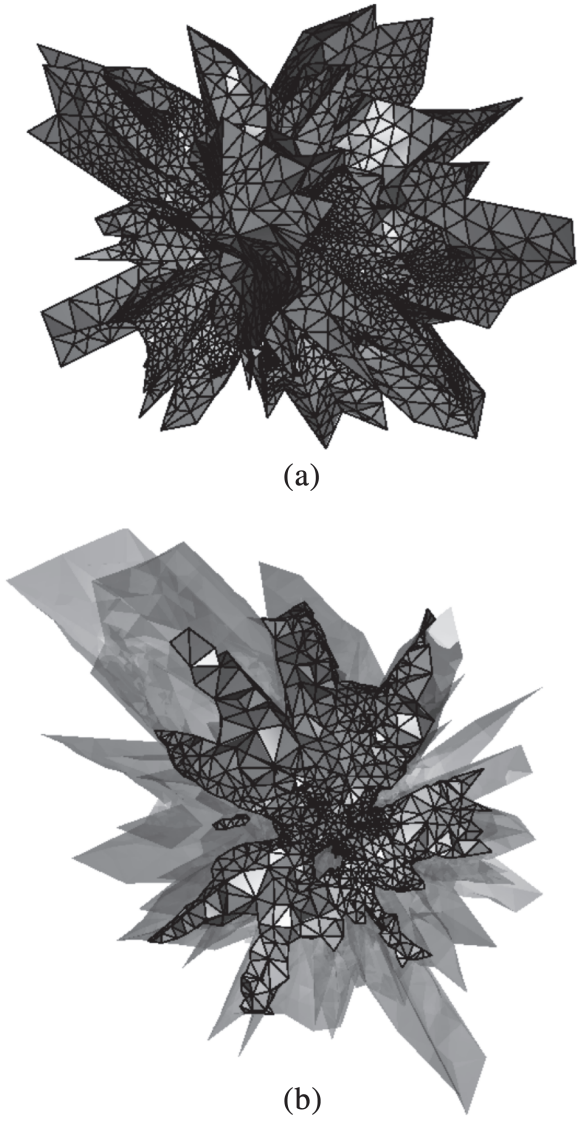

FIG. 1. Problem 1: crystal domain.

the standard interpolation norm $(L, M)_{1/2}$ and also for the simplified version $(L, I)_{1/2}$. We see indeed that mesh independence, as well as a reduced number of iterations is achieved, when the right scaling is included. This feature of the preconditioner has not been considered in the literature, with most interface preconditioning papers providing local definitions of the square root Laplacian on uniform meshes.

The performance of $(L, M)_{1/2}$ on other nontrivial three-dimensional domains is extensively investigated in Kourounis (2008). In particular, it is demonstrated that in three dimensions the preconditioner $(L, M)_\theta$ with θ chosen empirically in $(0.5, 0.8)$ (by performing benchmarks on the coarsest mesh level)

TABLE 14 *Iterations for the crystal problem with and without the mass matrix*

	$(L, M)_{1/2}$				$(L, I)_{1/2}$			
n	$N = 4$	$N = 16$	$N = 64$	$N = 256$	$N = 4$	$N = 16$	$N = 64$	$N = 256$
67,799	16	21	23	NA	112	160	159	NA
240,832	16	17	24	31	57	110	104	141
2,521,753	17	21	25	29	30	28	71	65

TABLE 15 *Inverse Lanczos approximation: additional FGMRES iterations* $\text{it}(n_B) - \text{it}(k)$

	$k = 10$				$k = 15$				$k = 20$			
N	4	16	64	256	4	16	64	256	4	16	64	256
Level = 1	1	0	0	0	1	0	0	0	1	0	0	1
2	0	−1	−2	−1	0	−1	−1	−1	0	−1	−1	0
3	2	−1	−3	−3	1	−1	−3	−2	0	−1	−1	−1

is robust also with respect to the number of subdomains. One can also find in Kourounis (2008) a study of the parallel performance of our preconditioning approach on different platforms and operating systems (Linux, Solaris, AIX).

4.3 *Lanczos approximation of $H^{1/2}$-norms*

We end our experiments with a numerical study of the Lanczos approximation of the matrix product $\widetilde{H}_{1/2}^{-1}\mathbf{z}$ for given $\mathbf{z}$. We chose to run the experiment corresponding to problem (4.1) on a uniform mesh using both the inverse Lanczos approximations with several choices of k for all the mesh configurations employed in the experiment. In this case each iteration requires the inversion of a one-dimensional Laplacian problem on the interface in order to compute the action of the inner product associated with this method. This is only slightly more expensive than the simple matrix–vector multiplication by M required by the direct approach. However, the additional cost is well justified as the performance of the method appears to be independent of the level of discretization and the subdomain discretization (or N), cf. Table 15. Note that some improvement over the exact case occurs.

Finally, we note that for large problems the dimension of the interface may require a preconditioned iterative method such as PCG for the inversion of the matrices arising in the Lanczos process (both direct and inverse). We note that the natural preconditioner in this case is a block-diagonal preconditioner corresponding to inverting (in parallel) the operators corresponding to each edge Γ_i in the interface Γ. A study of this approach, together with analysis of the Lanczos iterations for domain decomposition, will be performed elsewhere.

5. Conclusion

We presented an interface preconditioner arising from the coercivity and continuity properties of the boundary Schur complement. The preconditioner is a discrete fractional Sobolev norm corresponding to the finite element discretization of the norm $\|\cdot\|_{H_{00}^{1/2}(\Gamma)}$. Its application is achieved via sparse Lanczos procedures that do not change the complexity of subdomain solves. Our preconditioners do not require an overlapping procedure nor the calibration of any parameters. For the case of constant coefficients, the

analysis shows that the performance of the method is independent of the mesh parameter h. Numerically, we found that the performance depends sublinearly on the number of subdomains. For nonconstant coefficients our interface preconditioners generalize in a natural fashion to discrete norms corresponding to the interpolation of suitable weighted spaces. For this case we demonstrated that the performance is independent also of the coefficients. Three-dimensional extensions of this approach are straightforward and exhibit the same convergence properties.

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